What Practitioners Need To Know . . .

. . . About Optimization

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Optimization is a process by which we determine the most favorable tradeoff between competing interests, given the constraints we face. Within the context of portfolio management, the competing interests are risk reduction and return enhancement. Asset allocation is one form of optimization. We use an optimizer to identify the asset weights that produce the lowest level of risk for various levels of expected return. Optimization is also used to construct portfolios of securities that minimize risk in terms of tracking error relative to a benchmark portfolio. In these applications, we are usually faced with the constraint that the asset weights must sum to one.

We can also employ optimization techniques to manage strategies that call for offsetting long and short positions. Suppose, for example, that we wish to purchase currencies expected to yield high returns and to sell currencies expected to yield low returns, with the net result that we are neither long nor short the local currency. In this case, we would impose a constraint that the currency exposures sum to zero.

This column is intended as a tutorial on optimization. We will demonstrate, through the use of numerical examples, how to optimize a two-asset portfolio with only a pencil and the back of an envelope. If you wish to include three assets, you may need the front of the envelope as well. Beyond three assets, a computer would come in handy.

The Objective Function

Suppose we wish to identify the combinations of stocks and bonds that produce the lowest levels of risk for varying amounts of expected return. To begin, we must define a portfolio's expected return and risk.

The expected return of a portfolio comprised of just stocks and bonds is simply the weighted average of the assets' expected returns, as shown below:

Eq. 1

$$R_P = (R_S \cdot W_S) + (R_B \cdot W_B)$$

where

R_P = the portfolio's expected return;
R_S = the expected return of stocks;
R_B = the expected return of bonds;
W_S = the percentage of the portfolio allocated to stocks; and
W_B = the percentage allocated to bonds.

Portfolio risk is a little trickier. It is defined as volatility, and it is measured by the standard deviation or variance (the standard deviation squared) around the portfolio's expected return.[1] To compute a portfolio's variance, we must consider not only the variance of the component assets' returns, but also the extent to which the assets' returns co-vary.[2] The variance of a portfolio of stocks and bonds is computed as follows:

Eq. 2

$$V = (\sigma_S^2 \cdot W_S^2) + (\sigma_B^2 \cdot W_B^2) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_S)(\sigma_B \cdot W_B)$$

where

V = the portfolio variance;
σ_S = the standard deviation of stocks;
σ_B = the standard deviation of bonds; and
cor = the correlation between stocks and bonds.

Our objective, as stated earlier, is to minimize portfolio risk. Our first constraint is that the weighted average of the stock and bond returns must equal the expected return for the portfolio. We are also faced with a second constraint. We must allocate our entire portfolio to some combination of stocks and bonds. We would not want to leave part of it uninvested. Therefore, the fraction we allocate to stocks plus the fraction we allocate to bonds must equal 1.

We can combine our objective and constraints to form the following objective function:

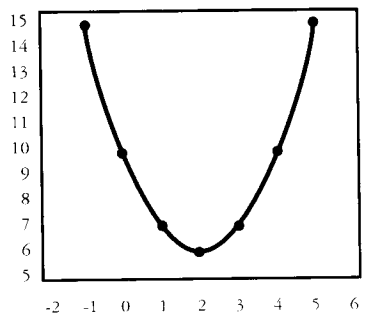
Eq. 3

$$\phi = [(\sigma_S^2 \cdot W_S^2) + (\sigma_B^2 \cdot W_B^2) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_S)(\sigma_B \cdot W_B)] + [\lambda_1(R_S \cdot W_S + R_B \cdot W_B - R_P)] + [\lambda_2(W_S + W_B - 1)]$$

The first two lines of Equation (3) simply equal portfolio variance, the quantity to be minimized. The third line represents the first constraint, which ensures that the weighted average of the stock and bond returns equals the portfolio's expected return. The Greek letter lambda (λ) is called a Lagrange multiplier. It is a variable introduced to facilitate optimization when we face constraints, and it does not always lend itself to economic interpretation. The fourth line represents the second

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Figure A A Function



constraint, which guarantees that the portfolio is fully invested. Again, lambda serves to facilitate a solution.

A Digression on Calculus

You may recall from elementary calculus that a function reaches its minimum or maximum value when its first derivative, or slope, equals 0. The first derivative measures the amount by which the value of the function changes given a one-unit change in the variable upon which it depends. Consider the following quadratic function:

$$y = x^2 - 4x + 10$$

If we set its derivative, $2x - 4$, equal to zero and solve for x, we find that this function reaches its minimum value when x equals 2. Figure A confirms this.

As Figure A shows, a function reaches its extreme value when it flattens out or has no slope, which is to say when its first derivative equals zero. With this insight, let us return to our objective function.

Unlike the quadratic function described above, which has a single unknown value, our objective function has four unknown values—the percentage of the portfolio to be allocated to stocks, the percentage to be allocated to bonds, the Lagrange multiplier for the first constraint and the Lagrange multiplier for the second constraint. To minimize portfolio risk given our constraints, we must take the partial derivative of the objective function with respect to each asset weight and with respect to each Lagrange multiplier and set it equal to zero, as shown below:

Eq. 4

$$\partial\phi/\partial W_S = 2(\sigma_S^2 \cdot W_S) + 2\,\mathrm{cor}\,(\sigma_S \cdot \sigma_B \cdot W_B) + \lambda_1 \cdot R_S + \lambda_2 = 0,$$

Eq. 5

$$\partial\phi/\partial W_B = 2(\sigma_B^2 \cdot W_B) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_S \cdot \sigma_B) + \lambda_1 \cdot R_B + \lambda_2 = 0,$$

Eq. 6

$$\partial\phi/\partial\lambda_1 = (R_S \cdot W_S) + (R_B \cdot W_B) - R_P = 0,$$

Eq. 7

$$\partial\phi/\partial\lambda_2 = W_S + W_B - 1 = 0.$$

Given assumptions for expected return, standard deviation and correlation (which we will identify later), we wish to find the values of W_S and W_B associated with different values of R_P, the portfolio's expected return. The values for λ_1 and λ_2 are merely mathematical byproducts of the solution.

One approach for solving a system of linear equations is by matrix inversion. Again, let us digress for a moment and focus on a simpler example of matrix inversion.[3]

A Digression on Matrix Algebra

In simple algebra, we would solve an expression such as $y = bx$ by dividing y by b. In matrix algebra, however, we cannot divide. The analogous operation is called matrix inversion.

Suppose we have the following system of linear equations:

$$6x_1 + 3x_2 + x_3 = 22$$

$$x_1 + 4x_2 - 2x_3 = 12$$

$$4x_1 - x_2 + 5x_3 = 10$$

We can represent this system of linear equations with a coefficient matrix, a vector for the variables and a vector for the constants, as follows:

$$\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

In order to solve for x_1, x_2 and x_3, we must find the inverse of the coefficient matrix and multiply it by the vector of constants. The inverse of the coefficient matrix is:[4]

$$\begin{bmatrix} 18/52 & -16/52 & -10/52 \\ -13/52 & 26/52 & 13/52 \\ -17/52 & 18/52 & 21/52 \end{bmatrix}$$

We multiply this inverse by the vector of constants to yield a vector with the values for x_1, x_2 and x_3, as follows

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18/52 & -16/52 & -10/52 \\ -13/52 & 26/52 & 13/52 \\ -17/52 & 18/52 & 21/52 \end{bmatrix} \cdot \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Now let us return to our system of linear equations and rewrite it in matrix notation.

$$\begin{bmatrix} 2 \cdot \sigma_S^2 & 2\mathrm{cor}\,(\sigma_S \cdot \sigma_B) & R_S & 1 \\ 2\,\mathrm{cor}\,(\sigma_S \cdot \sigma_B) & 2 \cdot \sigma_B^2 & R_B & 1 \\ R_S & R_B & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} W_S \\ W_B \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R_P \\ 1 \end{bmatrix}$$

We must next estimate the inputs for expected return, standard de-

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Table I Capital Market Assumptions

	Expected Return	*Standard Deviation*	*Correlation*
Stocks	12%	20%	
Bonds	8%	10%	50%

viation and correlation. Suppose that, after careful examination of historical data and thorough analysis of current economic and capital market conditions, we arrive at the assumptions shown in Table I.

With these assumptions, we can rewrite the coefficient matrix as follows:

$$\begin{bmatrix} 0.08 & 0.02 & 0.12 & 1.00 \\ 0.02 & 0.02 & 0.08 & 1.00 \\ 0.12 & 0.08 & 0.00 & 0.00 \\ 1.00 & 1.00 & 0.00 & 0.00 \end{bmatrix}$$

Its inverse equals:

$$\begin{bmatrix} 0 & 0 & 25 & -2 \\ 0 & 0 & -25 & 3 \\ 25 & -25 & -37.5 & 3 \\ -2 & 3 & 3 & -0.26 \end{bmatrix}$$

Because the constant vector includes a variable for the portfolio's expected return, we obtain a vector of formulas, rather than values, when we multiply the inverse matrix by the vector of constants, as shown below.

$$\begin{bmatrix} W_S \\ W_B \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 25 \cdot R_P - 2 \\ -25 \cdot R_P + 3 \\ -37.5 \cdot R_P + 3 \\ 3 \cdot R_P - 0.26 \end{bmatrix}$$

We are interested only in the first two formulas. The first formula yields the percentage to be invested in stocks in order to minimize risk when we substitute a value for the portfolio's expected return. The second formula yields the percentage to be invested in bonds. Table II shows the allocations to stocks and bonds that minimize risk for portfolio expected returns ranging from 8% to 13%.

Table II Optimal Portfolios

Expected Return	*Standard Deviation*	*Stock %*	*Bond %*
8.00%	10.00%	0	100
9.00%	10.90%	25	75
10.00%	13.23%	50	50
11.00%	16.39%	75	25
12.00%	20.00%	100	0
13.00%	23.85%	125	−25

From Table II we see that we must sell bonds short and lever our exposure to stocks in order to achieve a portfolio expected return that is greater than the higher of the expected return on stocks or bonds.

An Alternative Approach

In 1987, William Sharpe published an algorithm for portfolio optimization that has the dual virtues of accommodating many real-world complexities while appealing to our intuition.[5] Consider the following objective function to be maximized.

Eq. 8

$$\phi = R - \lambda \cdot V$$

where

R = portfolio expected return;
λ = a risk-aversion coefficient; and
V = portfolio variance.

The risk-aversion coefficient measures our willingness to sacrifice one unit of expected return in order to reduce risk (variance) by one unit. By maximizing this objective function, we maximize expected return minus a quantity representing our aversion to risk times risk (as measured by variance).

Assume, again, that we have a portfolio consisting of stocks and bonds. Substituting the equations for portfolio expected return and variance (Equations (1) and (2)), we can rewrite the objective function of Equation (8) as follows:

Eq. 9

$$(R_S \cdot W_S) + (R_B \cdot W_B) - \lambda[(\sigma_S^2 \cdot W_S^2) + (\sigma_B^2 \cdot W_B^2) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_S \cdot \sigma_B \cdot W_B)].$$

This objective function measures the utility or satisfaction we derive from a particular combination of expected return and risk, given our attitude toward risk. Its partial derivative with respect to each asset weight is:

Eq. 10

$$\partial\phi/\partial W_S = R_S - \lambda[2(\sigma_S^2 \cdot W_S) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_B \cdot \sigma_B)],$$

Eq. 11

$$\partial\phi/\partial W_B = R_B - \lambda[2(\sigma_B^2 \cdot W_B) + 2\,\mathrm{cor}\,(\sigma_S \cdot W_S \cdot \sigma_B)].$$

The equation for each asset measures how much we can increase or decrease our utility, starting from our current asset mix, by increasing our exposure to that asset. A negative partial derivative indicates that we can improve utility by reducing exposure to that asset, while a positive partial derivative indicates that we should raise the exposure to that asset in order to improve our utility.

Let us retain our earlier assumptions about the expected returns and standard deviations of stocks and bonds and their correlation. Further, let us assume that our portfolio is currently allocated 60% to stocks and 40% to bonds and that our aversion toward risk equals 2. A risk-aversion coefficient of 2 means that we are willing to reduce expected return by two units in order to lower variance by one unit.

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If we substitute these values into the partial derivative equations, we find that we can improve our utility by 0.008 units if we increase our exposure to stocks by 1%, and that we can improve our utility by 0.04 units if we increase our exposure to bonds by 1%. Both derivatives are positive. However, we can only allocate 100% of the portfolio. We should therefore increase our exposure to the asset class with the higher derivative by 1% and reduce by the same amount our exposure to the asset class with the lower derivative. In this way, we will ensure that we are always 100% invested.

Having switched our allocations in line with the relative magnitudes of the derivatives, we recompute the partial derivatives given our new allocation of 59% stocks and 41% bonds. Again, bonds have a higher derivative than stocks; hence we shift again from stocks to bonds. If we proceed in this fashion, we will find that when our portfolio is allocated 1/3 to stocks and 2/3 to bonds, the derivatives are exactly equal to each other. At this point, we cannot improve our utility any further by changing the allocation between stocks and bonds. We have maximized our objective function.

By varying the values we assign λ, we can identify mixes of stocks and bonds for many levels of risk-aversion, thus enabling us to construct the entire efficient frontier of stocks and bonds.

By now you are probably wondering why you should bother with fancy computer programs to optimize portfolios, since you can optimize them on the back of an envelope. We have purposely dealt with only two assets. As we increase the number of assets under consideration, we increase by a faster rate the number of variables to be included in our objective function. The sum of the number of expected returns, standard deviations and correlations to be included equals $(n^2 + n)/2$, where n equals the number of assets we wish to consider. Moreover, our examples are abstractions of real-world optimization problems. In the real world, we may be faced with short-selling constraints and other allocation constraints, as well as with transaction costs. Although it is certainly useful to understand the intuition and mathematical techniques that underlie optimization, most practical applications require sophisticated computer programs.

Error Maximization

There is a dark side to optimization, which we must not ignore. Cynics refer to it as "error maximization." In search of assets that increase a portfolio's expected return and lower its risk, optimization, by its vary nature, favors assets for which expected return is overestimated and risk underestimated.

Consider, for example, two assets, both of which have true mean returns of 10%. We do not know the true mean returns, however. We must estimate them, and in so doing we are vulnerable to errors. Suppose we estimate one asset to have a mean return of 8% and the other to have a mean return of 12%. If they both contribute the same degree of risk to the portfolio, the optimizer will favor the asset with the 12% return estimate.

This problem pertains as well to errors in our risk estimates. Moreover, the problem is exacerbated as we increase the number assets. The errors do not cancel each other out. Rather, they accumulate.

This bias leads to two problems. First, the expected return of the optimized portfolio is overstated, while its risk is understated. We can redress this problem to some extent by reducing our portfolio's estimated expected return or by augmenting its estimated risk to account for estimation error.

There is a more pernicious consequence of error maximization, though. Not only do we misestimate the portfolio's expected return and risk, we do so for the wrong portfolio. Essentially, we have excluded or underweighted assets for which we have underestimated return and overestimated risk, substituting for them assets for which we have erred in the opposite direction.

A common palliative for error maximization is to force the estimates to converge to their cross-sectional mean. We simply blend this mean with each individual estimate. For example, we might construct a new estimate for an asset's expected return by weighting the mean of all the expected return estimates by 1/3 and the original estimate for that particular asset by 2/3. This approach is limited, however, because it assumes that all our estimates are equally susceptible to error.

As an alternative, we might try to scale our estimates based on whatever information we can gather about their reliability. If we derive our estimates by extrapolating historical data, for example, we might wish to scale them as a function of the number of observations we used. At the very least, we should be sensitive to error maximization as we interpret our optimization results.

Footnotes

1. *For a more detailed discussion of risk, see M. Kritzman, "What Practitioners Need to Know About Uncertainty,"* Financial Analysts Journal *March/April 1991.*
2. *The notion of covariance is discussed in M. Kritzman, "What Practitioners Need to Know About the Nobel Prize,"* Financial Analysts Journal *January/February 1991.*
3. *The inverse of a matrix is analogous to a reciprocal in simple algebra. If we multiply a matrix by its inverse, we get an identity matrix. An identity matrix includes ones along its diagonal and zeros for all the other elements. It is analogous to the number one in simple algebra, in that a*

Concluded on page 18.

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